CURVATURE STRUCTURE OF SELF-DUAL 4-MANIFOLDS

NOVICA BLAŽIĆ, PETER GILKEY, STANA NIKČEVIĆ, AND IVA STAVROV

ABSTRACT. We show the existence of a modified $\mathrm{Cliff}(1,1)$-structure compatible with an Osserman 0-model of signature $(2,2)$. We then apply this algebraic result to certain classes of pseudo-Riemannian manifolds of signature $(2,2)$. We obtain a new characterization of the Weyl curvature tensor of an (anti-)self-dual manifold and we prove some new results regarding (Jordan) Osserman manifolds.

DEDICATION

This paper is one of several projects that were begun by Novica Blažić but not completed owing to his untimely death in 2005. The work has been finished to preserve his mathematical legacy and is dedicated to his memory.

1. INTRODUCTION

Let R be the curvature operator of the Levi-Civita connection of a pseudo-Riemannian manifold M. The *Jacobi operator* $J_R(x) : T_P M \to T_P M$ corresponding to unit spacelike or unit timelike tangent vectors x is characterized by

$$J_R(x)y := R(y, x)x, \quad x \in T_P M.$$

It plays a central role in curvature theory [4, 17].

It was conjectured by Osserman [22] in the Riemannian setting that the spectrum of the Jacobi operator $J_R(x)$ is independent of the choice of a unit tangent vector x and its base point P if and only if the underlying Riemannian manifold is locally rank 1-symmetric or flat. This conjecture follows from work of Chi [8] and Nikolayevsky [19, 20, 21] in dimensions other than 16; this question is still open in dimension 16. Nikolayevsky executed the approach outlined in [18], the major part of which is in showing that an Osserman 0-model (see the next section for the definition) allows a compatible Clifford algebra structure. This crucial step is the algebraic counterpart to studying the so-called *point-wise Osserman manifolds*: manifolds where the spectrum of the Jacobi operator $J_R(x)$ does not depend on the choice of the unit tangent vector x at any of the base points, but is allowed to vary from point to point. Note that there exists a nice connection between point-wise Osserman and self-dual Einstein Riemannian manifolds due to the work of Sekigawa and Vanhecke [23]. There are many other properties of the curvature operator which can be studied similarly – see, for example, [11, 16].

Pseudo Riemannian and conformal geometry is central to many investigations and the phenomena are often very different from the Riemannian setting (see,

for example, [5, 9]), and the study of Osserman manifolds is no exception. In the pseudo-Riemannian geometry the study of Osserman-type manifolds becomes rather complicated because their Jacobi operators need not be diagonalizable. We say a pseudo-Riemannian manifold is (point-wise) timelike/spacelike Jordan Osserman if the Jordan normal form of $J_R(x)$ is independent of the choice of unit timelike/spacelike vector x. Numerous examples have been constructed [6] which show the existence of non-homogeneous point-wise Jordan Osserman manifolds.

The Weyl curvature tensor W, which depends only on the conformal class of a pseudo-Riemannian manifold, obeys the same algebraic symmetries as the Riemann curvature tensor itself (see the next section for details). Quite naturally one is led to investigating the spectral geometry of the *conformal Jacobi operator*

$$J_W(x)y = W(y, x)x.$$

We say that a pseudo-Riemannian manifold is *conformally Osserman* if for each base point the spectrum (or equivalently the characteristic polynomial) of the conformal Jacobi operator $J_W(x)$ is independent of the choice of the unit tangent vector x. Note that the spectrum is allowed to vary from point to point. As in the case of point-wise Osserman manifolds, conformally Osserman and self-dual manifolds of dimension 4 are closely related. More precisely, we have the following theorem (see also [2],[6]).

Theorem 1.1. *A 4-dimensional oriented pseudo-Riemannian manifold is conformally Osserman if and only if it is self-dual or anti-self-dual.*

It should be pointed out that in the positive definite setting in dimension 4 one can actually show point-wise existence of a Clifford algebra structure compatible with a (conformal) Osserman algebraic curvature tensor [2]. More precisely, for a conformal Osserman algebraic curvature tensor R there exist skew-adjoint operators $\Phi_1, \Phi_2, \Phi_3 = \Phi_2\Phi_1$ and constants $\lambda_1, \lambda_2, \lambda_3$ such that

- $\Phi_i\Phi_j + \Phi_j\Phi_i = -2\delta_{ij}\,\mathrm{Id}$;
- $\lambda_1 + \lambda_2 + \lambda_3 = 0$;
- $W = \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3}$, where

(1.a) $R_\Phi(x, y)z := g(\Phi y, z)\Phi x - g(\Phi x, z)\Phi y - 2g(\Phi x, y)\Phi z.$

The tensors R_Φ naturally appear in the geometry of rank 1-symmetric spaces: if we let

$$R_0(x, y)z = g(x, z)y - g(y, z)x$$

denote the Riemann curvature tensor of the standard sphere, then the curvature tensor of the projective spaces $\mathbb{C}P^n$ and $\mathbb{H}P^n$ can be written as

$$R_0 + R_I \quad \text{and} \quad R_0 + R_I + R_J + R_K, \quad \text{respectively.}$$

Here I and $\{I, J, K\}$ are the canonical complex and quaternion structures on $\mathbb{C}P^n$ and on $\mathbb{H}P^n$, respectively. Operators of this type also are central to the analysis of [10].

In this paper we will primarily study Osserman 0-models of signature $(2, 2)$; they are a convenient algebraic abstraction of what is happening (point-wise) on Osserman and conformal Osserman manifolds (of signature $(2,2)$). Their exact definitions and other preliminaries can be found in Section 2. Section 3 is dedicated to the proof of the following proposition, which is our main algebraic result; this result plays a crucial role in the analysis of [7, 12].

Proposition 1.2. *A 0-model $\mathfrak{M} = (V, g, A)$ of signature $(2, 2)$ is Osserman if and only if there exist skew-adjoint linear operators $\Phi_1, \Phi_2, \Phi_3 = \Phi_2\Phi_1$ and constants λ_i, λ_{ij} such that*

- $\Phi_1^2 = -\operatorname{Id}, \Phi_2^2 = \Phi_3^2 = \operatorname{Id}$;
- $\Phi_i\Phi_j + \Phi_j\Phi_i = 0$ *if* $i \neq j$;
- $A = \lambda_0 R_0 + \sum \lambda_i R_{\Phi_i} + \sum_{i<j} \lambda_{ij}[R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)}]$.

A triple of skew-adjoint operators (Φ_1, Φ_2, Φ_3), $\Phi_3 = \Phi_2\Phi_1$, satisfying the first two identities of the previous proposition is often referred to as a $\operatorname{Cliff}(1, 1)$-*structure on* (V, g). A very interesting aspect of the tensor decomposition in the third identity is the appearance of the tensor $R_{(\Phi_i - \Phi_j)}$. It is particularly interesting that for some choices of i, j we have $(\Phi_i - \Phi_j)^2 = 0$. Due to this distinct feature we will say that A admits a *modified* $\operatorname{Cliff}(1, 1)$-*structure*. It should also be pointed out that the usage of the tensor R_0 is optional (see Remark 3.3 below).

Proposition 1.2 can be applied to the geometric setting. In Section 4 we discuss the following geometric results. As in Proposition 1.2 the usage of the tensor R_0 is optional.

Theorem 1.3. *A pseudo-Riemannian manifold of signature $(2, 2)$ is point-wise Osserman if and only if for each point of the manifold there exists a local smooth $\operatorname{Cliff}(1, 1)$-structure (Φ_1, Φ_2, Φ_3) and smooth functions λ_i, λ_{ij} such that*

$$R = \lambda_0 R_0 + \sum \lambda_i R_{\Phi_i} + \sum_{i<j} \lambda_{ij}[R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)}].$$

In the light of the Theorem 1.1 we have the following characterization of the Weyl curvature of (anti-)self-dual manifolds.

Theorem 1.4. *Let (M, g) be an oriented manifold of signature $(2, 2)$. The following conditions are equivalent.*

(1) *M is conformally Osserman;*
(2) *M is self-dual or anti-self-dual;*
(3) *For each $P \in M$ there exists a local smooth $\operatorname{Cliff}(1, 1)$-structure (Φ_1, Φ_2, Φ_3) and smooth functions λ_i, λ_{ij} such that $\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3 = 0$ and*

$$W = \lambda_0 R_0 + \sum \lambda_i R_{\Phi_i} + \sum_{i<j} \lambda_{ij}[R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)}].$$

The non-homogeneous examples of (conformally) Osserman manifolds given in [6] indicate that in general one can not find a $\operatorname{Cliff}(1, 1)$-structure (Φ_1, Φ_2, Φ_3) such that the corresponding functions λ_i, λ_{ij} are constant. Indeed, if one could find such Φ_i the Jordan normal form of the (conformal) Jacobi operators would have to be independent of the base point, contrary to the examples of [6]. A natural question at this point is if the (Weyl) curvature tensor of a *globally (conformally) Jordan Osserman manifold*, i.e. (conformally) Osserman manifold whose Jordan normal form of the (conformal) Jacobi operator is independent of the base point, allows a decomposition with constant functions λ_i, λ_{ij}. An affirmative answer to this question is proven in Section 4.

Theorem 1.5. *If a connected pseudo-Riemannian manifold of signature $(2, 2)$ is globally Jordan Osserman (resp. globally conformally Jordan Osserman) then the $\operatorname{Cliff}(1, 1)$-structure of Theorem 1.3 (resp. Theorem 1.4) can be chosen so that the functions λ_i, λ_{ij} are constant.*

2. Preliminaries

In what follows we will assume (M, g) is a 4-dimensional pseudo-Riemannian manifold of neutral signature $(2, 2)$. When referring to a pseudo-orthonormal frame or a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ we will always assume

$$g(e_i, e_j) = \epsilon_i \delta_{ij}, \quad \text{where} \quad \epsilon_1 = \epsilon_2 = -1, \quad \epsilon_3 = \epsilon_4 = 1.$$

The Riemann curvature tensor R of (M, g) satisfies the following symmetries:

(2.a) $R(x, y, z, v) = -R(y, x, z, v) = -R(x, y, v, z),$

(2.b) $R(x, y, z, v) = R(z, v, x, y),$ and

(2.c) $R(x, y, z, v) + R(y, z, x, v) + R(z, x, y, v) = 0$

The curvature tensor R restricted to a tangent space $T_P M$ is an example of an *algebraic curvature tensor:* a 4-tensor on an innerproduct space which satisfies symmetries (2.a)-(2.c). This abstract setting is convenient when working with point-wise properties of the geometric curvature tensor. The triple $\mathfrak{M} = (V, g, A)$, where A is an algebraic curvature tensor on the innerproduct space (V, g), is called a 0-*model* [14].

Another important example of a 0-model comes from conformal geometry. The *Weyl tensor W* is obtained from the decomposition

(2.d) $$R = \frac{Scal}{24} \, g \cdot g + \frac{1}{2}\Big(Ric - \frac{Scal}{4} \, g\Big) \cdot g + W,$$

where Ric denotes the Ricci curvature (the contraction of R with respect to the first and the third slots), $Scal$ denotes the scalar curvature (the contraction of Ric) and where $h \cdot k$ deonotes the Kulkarni-Nomizu product of two symmetric 2-tensors:

$$\begin{aligned} h \cdot k \ (v_1, v_2, v_3, v_4) \ &= h(v_1, v_3)k(v_2, v_4) + h(v_2, v_4)k(v_1, v_3) \\ &\quad -h(v_1, v_4)k(v_2, v_3) - h(v_2, v_3)k(v_1, v_4). \end{aligned}$$

The Weyl tensor depends only on the conformal class of (M, g). Morover, it satisfies the curvature symmetries (2.a)-(2.c) and so we can treat it abstractly as an algebraic curvature tensor. In fact, we may use the decomposition (2.d) to associate the Weyl tensor W_A to any 0-model (V, g, A). Note that the Weyl tensor is always Ricci flat.

An algebraic curvature tensor A gives rise to the *Jacobi operator*, a family of operators $J_A(x)$ defined by

$$g(J_A(x)y, z) = A(y, x, x, z).$$

It follows from the curvature symmetries (2.a)-(2.c) that each $J_A(x)$, $x \neq 0$ induces a self-adjoint operator on the orthogonal complement $\{x\}^\perp$. In particular, for unit timelike vectors x in a vector space of signature $(2, 2)$ the operator $J_A(x)$ may be viewed as a self-adjoint operator on a vector space of signature $(1, 2)$.

Following the terminology of the spectral geometry of the Riemann curvature tensor, we say that a 0-model (V, g, A) is *Osserman* (resp. *conformal Osserman*) if the characteristic polynomial of $J_A(x)$ (resp. $J_{W_A}(x)$) does not depend on the choice of timelike unit vector x. If A is (conformal) Osserman then the characteristic polynomial of the Jacobi operator does not depend on the choice of unit spacelike vector x either, see [13]. Osserman 0-models of signature $(2, 2)$ have been classified based upon the form of the corresponding minimal polynomial [1, 3, 6]. We have the following four types.

Theorem 2.1. *A 0-model (V, g, A) is Osserman if and only if one of the following holds.*

- TYPE I: *The Jacobi operators $J_A(x)$, $\|x\|^2 = -1$, are diagonalizable, i.e. have matrix representations of the form*

(2.e)
$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}.$$

  In this case if $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis with respect to which $J_A(e_1)$ has the matrix representation as above, then the non-vanishing components of A with respect to $\{e_1, e_2, e_3, e_4\}$ are:

$$A_{1221} = A_{4334} = -\alpha, \ A_{1331} = A_{4224} = \beta, \ A_{1441} = A_{3223} = \gamma,$$
$$A_{1234} = \frac{2\alpha - \beta - \gamma}{3}, \ A_{1423} = \frac{-\alpha - \beta + 2\gamma}{3}, \ A_{1342} = \frac{-\alpha + 2\beta - \gamma}{3}.$$

- TYPE II: *The Jacobi operators $J_A(x)$, $\|x\|^2 = -1$, have matrix representations of the form*

(2.f)
$$\begin{bmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \ \beta \neq 0.$$

  In this case if $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis with respect to which $J_A(e_1)$ has the matrix representation as above, then the non-vanishing components of A with respect to $\{e_1, e_2, e_3, e_4\}$ are:

$$A_{1221} = A_{4334} = -\alpha, \ A_{1331} = A_{4224} = \alpha, \ A_{1441} = A_{3223} = \gamma,$$
$$A_{2113} = A_{2443} = -\beta, \ A_{1224} = A_{1334} = \beta,$$
$$A_{1234} = \frac{\alpha - \gamma}{3}, \ A_{1423} = \frac{2(\gamma - \alpha)}{3}, \ A_{1342} = \frac{\alpha - \gamma}{3}.$$

- TYPE III: *The Jacobi operators $J_A(x)$, $\|x\|^2 = -1$, have matrix representations of the form*

(2.g)
$$\begin{bmatrix} \epsilon(\alpha - \frac{1}{2}) & \epsilon\frac{1}{2} & 0 \\ -\epsilon\frac{1}{2} & \epsilon(\alpha + \frac{1}{2}) & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \epsilon = \pm 1, \ \alpha, \beta, \in \mathbb{R}.$$

  In this case if $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis with respect to which $J_A(e_1)$ has the matrix representation as above, then the non-vanishing components of A with respect to $\{e_1, e_2, e_3, e_4\}$ are:

$$A_{1221} = A_{4334} = -\epsilon(\alpha - \frac{1}{2}), \ A_{1331} = A_{4224} = \epsilon(\alpha + \frac{1}{2}),$$
$$A_{1441} = A_{3223} = \beta,$$
$$A_{2113} = A_{2443} = -\epsilon\frac{1}{2}, \ A_{1224} = A_{1334} = \epsilon\frac{1}{2},$$
$$A_{1234} = \frac{\epsilon(\alpha - \frac{3}{2}) - \beta}{3}, \ A_{1423} = \frac{-2\epsilon\alpha + 2\beta}{3}, \ A_{1342} = \frac{\epsilon(\alpha + \frac{3}{2}) - \beta}{3}.$$

- TYPE IV: *The Jacobi operators $J_A(x)$, $\|x\|^2 = -1$, have matrix representations of the form*

(2.h)
$$\begin{bmatrix} \alpha & 0 & \frac{\sqrt{2}}{2} \\ 0 & \alpha & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \alpha \end{bmatrix}, \quad \alpha \in \mathbb{R}.$$

In this case if $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis with respect to which $J_A(e_1)$ has the matrix representation as above, then the non-vanishing components of A with respect to $\{e_1, e_2, e_3, e_4\}$ are:

$$A_{1221} = A_{4334} = -\alpha, \ A_{1331} = A_{4224} = \alpha, \ A_{1441} = A_{3223} = \alpha,$$

$$A_{2114} = A_{2334} = -\frac{\sqrt{2}}{2}, \ A_{3114} = -A_{3224} = \frac{\sqrt{2}}{2},$$

$$A_{1223} = A_{1443} = A_{1332} = -A_{1442} = \frac{\sqrt{2}}{2}.$$

Here is an important corollary of this classification result.

Corollary 2.2. *Let A and $\widetilde{A}$ be two Osserman algebraic curvature tensors on an innerproduct space (V, g) of signature $(2, 2)$. If for some timelike unit vector x we have $J_A(x) = J_{\widetilde{A}}(x)$, then necessarily $A = \widetilde{A}$.*

As mentioned in the Introduction, (Jordan) Osserman algebraic curvature tensors tend to be related to representations of Clifford algebras. We proceed by investigating this relationship in signature $(2, 2)$.

3. Clifford structures and Proposition 1.2

Let $\left(\mathbb{R}^{(p,q)}, (.,.)\right)$ denote the standard innerproduct space of signature (p, q). The Clifford algebra $\mathrm{Cliff}(p, q)$ is the unital algebra generated by $\mathbb{R}^{(p,q)}$ subject to the Clifford commutation relations:

$$v \cdot w + w \cdot v = -2(v, w) \cdot 1.$$

The Clifford algebra $\mathrm{Cliff}(0, 2)$, for example, can be seen as the algebra of quaternions $\mathbb{H} := \mathrm{Span}_{\mathbb{R}}\{1, i, j, k\}$, where

$$i^2 = j^2 = k^2 = -1, \quad ij + ji = ik + ki = jk + kj = 0, \quad ijk = -1.$$

Note that the multiplication on the left by i, j and k gives rise to a unitary representation of $\mathrm{Cliff}(0, 2)$ on $\mathbb{R}^4$. Likewise, $\mathrm{Cliff}(1, 1) = \mathrm{Span}_{\mathbb{R}}\{I, J, K\}$ are the para-quaternions; these satisfy the relations

$$I^2 = -1, \quad J^2 = K^2 = 1, \quad IJ + JI = IK + KI = JK + KI = 0, \quad IJK = 1.$$

The main results of our paper rely on the existence of the following representation of $\mathrm{Cliff}(1, 1)$ on $\mathbb{R}^{(2,2)}$.

Lemma 3.1. *Let $\{e_1, e_2, e_3, e_4\}$ be a pseudo-orthonormal basis for an innerproduct space (V, g) of signature $(2, 2)$. There exist skew-adjoint linear maps $\Phi_i : V \to V$, where $i = 1, 2, 3$, such that*

1. $\Phi_i \Phi_j + \Phi_j \Phi_i = 0, \ i \neq j$;
2. $\Phi_1^2 = -\mathrm{Id}, \ \Phi_2^2 = \Phi_3^2 = \mathrm{Id}$;
3. $\Phi_3 = \Phi_2 \Phi_1$;
4. $\Phi_1(e_1) = e_2, \ \Phi_2(e_1) = e_3, \ \Phi_3(e_1) = e_4$.

Proof. Our choice of orthonormal basis allows us to identify V with $\mathbb{R}^{(1,1)} \otimes \mathbb{R}^{(0,2)}$. More precisely, there exists an isometry $T : V \to \mathbb{R}^{(1,1)} \otimes \mathbb{R}^{(0,2)}$ with

$$T(e_1) = (1,0) \otimes (1,0), \quad T(e_2) = (1,0) \otimes (0,1),$$
$$T(e_3) = (0,1) \otimes (1,0), \quad T(e_4) = (0,1) \otimes (0,1).$$

Consider

$$\alpha_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \alpha_1 \alpha_0;$$

these matrices satisfy $\alpha_i \alpha_j + \alpha_j \alpha_i = 0$ if $i \neq j$, $\alpha_0^2 = \alpha_1^2 = \text{Id}$ and $\alpha_2^2 = -\text{Id}$. Viewed as operators on $\mathbb{R}^{(1,1)}$ α_0 and α_2 are self-adjoint while α_1 is skew-adjoint. It is now easy to check that Φ_i, $i = 1, 2, 3$, defined by

$$T \circ \Phi_1 \circ T^{-1} = \alpha_0 \otimes \alpha_2, \quad T \circ \Phi_2 \circ T^{-1} = \alpha_1 \otimes \text{Id}, \quad T \circ \Phi_3 \circ T^{-1} = \alpha_2 \otimes \alpha_2$$

satisfy conditions (1)-(3). We note that:

$$\text{Id}(1,0) = \alpha_0(1,0) = (1,0), \quad \alpha_1(1,0) = \alpha_2(1,0) = (0,1).$$

Property (4) now follows. $\qquad\square$

Gilkey and Ivanova [15] gave a construction using Clifford algebras that showed the Jordan normal form of a Jordan Osserman algebraic curvature tensor can be arbitrary. We use their construction in what follows. Note that it follows from Lemma 3.1 that the map Φ_1 is an isometry and that the maps Φ_2, Φ_3 are anti-isometries:

$$g(\Phi_2 v, \Phi_2 w) = g(\Phi_3 v, \Phi_3 w) = -g(v, w).$$

Therefore, for unit timelike vector x the set $\{x, \Phi_1 x, \Phi_2 x, \Phi_3 x\}$ forms a pseudo-orthonormal basis.

Lemma 3.2. *Let Φ_i, $i = 1, 2, 3$, be skew-adjoint maps on an innerproduct space (V, g) of signature $(2, 2)$ satisfying relations (1) and (2) of Lemma 3.1. Let λ_i and λ_{ij} with $i < j$ be real constants. Then the curvature tensor*

$$R := \lambda_0 R_0 + \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3} + \sum_{i<j} \lambda_{ij} \big[R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)} \big]$$

gives rise to an Osserman 0-model on V.

Proof. In the computation which follows we will use π_x to denote the linear map

$$\pi_x(v) := g(v, x)x.$$

The Jacobi operator corresponding to an algebraic curvature tensor of the form R_Φ (see (1.a)) takes the form

$$J_{R_\Phi}(x)y = -3g(\Phi y, x)\Phi x = 3g(y, \Phi x)\Phi x, \quad \text{i.e.} \quad J_{R_\Phi}(x) = 3\pi_{\Phi x}.$$

The matrix representations of the operators

$$J_i(x) := J_{R_{\Phi_i}}(x) = 3\pi_{\Phi_i x}, \quad J_{ij}(x) := J_{R_{(\Phi_i - \Phi_j)}}(x) = 3\pi_{(\Phi_i x - \Phi_j x)}$$

with respect to $\{x, \Phi_1 x, \Phi_2 x, \Phi_3 x\}$ are independent of the choice of unit timelike vector x. Therefore,

$$\text{(3.a)} \quad J_R(x) = \lambda_0(\pi_x + \text{Id}) + 3\sum_{i=1}^{3} \lambda_i \pi_{\Phi_i x} + 3\sum_{i<j} \lambda_{ij} \big[\pi_{\Phi_i x} + \pi_{\Phi_j x} - \pi_{(\Phi_i x - \Phi_j x)}\big]$$

has its matrix representation with respect to $\{x, \Phi_1 x, \Phi_2 x, \Phi_3 x\}$ independent of the choice of a timelike unit vector x. So, the algebraic curvature tensor R is Osserman. $\qquad \square$

We can explicitly write down the matrix representation of the operator $J_R(x)$ (see equation (3.a)) with respect to the basis $\{x, \Phi_1 x, \Phi_2 x, \Phi_3 x\}$. To do so note that the operator $\pi_{\Phi_i x} + \pi_{\Phi_j x} - \pi_{(\Phi_i x - \Phi_j x)}$ is zero on $\mathrm{Span}\{\Phi_i x, \Phi_j x\}^\perp$, while it acts as

$$\Phi_i x \mapsto \epsilon_i \Phi_j x, \qquad \Phi_j x \mapsto \epsilon_j \Phi_i x$$

on $\mathrm{Span}\{\Phi_i x, \Phi_j x\}$. Hence our matrix representation is of the form

(3.b)
$$\begin{bmatrix} \lambda_0 - 3\lambda_1 & 3\lambda_{12} & 3\lambda_{13} \\ -3\lambda_{12} & \lambda_0 + 3\lambda_2 & 3\lambda_{23} \\ -3\lambda_{13} & 3\lambda_{23} & \lambda_0 + 3\lambda_3 \end{bmatrix}.$$

The proof of Proposition 1.2 is based upon the previous two lemmas and the classification of Theorem 2.1.

Proof of Proposition 1.2. Consider a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for (V, g) and consider maps Φ_i, $i = 1, 2, 3$ of Lemma 3.1; we have

$$e_2 = \Phi_1(e_1), \quad e_3 = \Phi_2(e_1), \quad e_4 = \Phi_3(e_1).$$

The matrix representation of the Jacobi operator $J_A(e_1)$ with respect to $\{e_2, e_3, e_4\}$ is of the form

$$J_A(e_1) = \begin{bmatrix} a & b & c \\ -b & d & e \\ -c & e & f \end{bmatrix},$$

where $a = -A_{2112}$, $b = -A_{3112}$, $c = -A_{4112}$, $d = A_{3113}$, $e = A_{4113}$ and $f = A_{4114}$. For a suitable choice of λ's the matrix (3.b) reduces to matrix above. To be precise, we need

(3.c) $\quad \lambda_1 = \dfrac{\lambda_0 - a}{3}, \ \lambda_2 = \dfrac{d - \lambda_0}{3}, \ \lambda_3 = \dfrac{f - \lambda_0}{3}, \ \lambda_{12} = \dfrac{b}{3}, \ \lambda_{13} = \dfrac{c}{3}, \ \lambda_{23} = \dfrac{e}{3}.$

Let λ_i, λ_{ij} be chosen as in (3.c) and let

(3.d) $\quad R := \lambda_0 R_0 + \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3} + \displaystyle\sum_{i<j} \lambda_{ij} \left[R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)} \right].$

It follows from the previous lemma that R is Osserman with $J_R(e_1) = J_A(e_1)$. We now use Corollary 2.2 to conclude that $R = A$. $\qquad \square$

Remark 3.3. *The constant λ_0 from the previous proof remains undetermined. Using (3.c) we see that $0 = \lambda_0 R_0 + \frac{\lambda_0}{3} R_{\Phi_1} - \frac{\lambda_0}{3} R_{\Phi_2} - \frac{\lambda_0}{3} R_{\Phi_3}$ for all λ_0 i.e.*

$$3R_0 = -R_{\Phi_1} + R_{\Phi_2} + R_{\Phi_3}.$$

Moreover, we can always set $\lambda_0 = 0$ and eliminate the R_0 term from the decomposition (3.d). On the other hand, the modified Clifford terms $R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)}$ are unavoidable in the cases when the conformal Jacobi operator is non-diagonalizable.

Proposition 1.2 can be applied to the Weyl tensor W_A of a conformal Osserman 0-model. Note though that the Ricci flatness of W_A imposes certain conditions on the constants λ_i. A short computation shows that the Ricci tensor corresponding to (3.d) is $Ric = 3(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3)g$; this means we should restrict our attention to constants λ_i with $\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3 = 0$.

Corollary 3.4. *A* 0-*model* $\mathfrak{M} = (V, g, A)$ *of signature* $(2, 2)$ *is conformal Osserman if and only if the Weyl tensor* W_A *allows a modified* $\mathrm{Cliff}(1, 1)$-*structure*

$$W_A = \lambda_0 R_0 + \sum \lambda_i R_{\Phi_i} + \sum_{i<j} \lambda_{ij}[R_{\Phi_i} + R_{\Phi_j} - R_{(\Phi_i - \Phi_j)}],$$

with $\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3 = 0$.

4. Local geometry

If the Riemann curvature tensor at a point $P \in M$ is Osserman, we say that the manifold (M, g) is *Osserman at* P; pseudo-Riemannian manifolds which are Osserman at each of their points are called *point-wise Osserman*. If the Weyl tensor of a pseudo-Riemannian manifold (M, g) is Osserman then M is said to be *conformally Osserman*. The first examples of (point-wise) Osserman and conformal Osserman manifolds of signature $(2, 2)$ are locally isotropic spaces $\mathbb{R}^{(2,2)}$, $S^{(2,2)}$ and $\mathbb{C}P^{(1,1)}$. The para-complex projective space $\widetilde{\mathbb{C}}P^2$ [13] is another example of an Osserman and conformally Osserman manifold.

The curvature decomposition of Proposition 1.2 carries over to the geometric setting, resulting in Theorems 1.3 and 1.4. To see this one needs to look back at the proof of Proposition 1.2 and replace the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with a smooth local pseudo-orthonormal frame. The functions λ_i and λ_{ij} in this case are smooth because they are given in terms of the components of the curvature tensor.

The really interesting examples of (conformally) Osserman manifolds come from manifolds having a parallel degenerate distribution of rank 2: the Walker manifolds [6, 14]. These manifolds provide a wide family of (conformally) Osserman manifolds which are not homogeneous. More specifically, we have the existence of (conformally) Osserman manifolds whose (conformal) Jacobi operator changes its Jordan normal form from point to point. Therefore, the decomposition of the Proposition 1.2 with *constant* λ's cannot be extended to the geometric setting without imposing further restrictions on the (conformal) Jacobi operators.

We say that a pseudo-Riemannian manifold is *globally Jordan Osserman (resp. globally conformally Jordan Osserman)* if the Jacobi operator $J_R(x)$ (resp. conformal Jacobi operator $J_W(x)$) has its Jordan normal form independent of the choice of unit timelike tangent vector x and its base point. To prove Theorem 1.5, the curvature decomposition result in this setting, we need a more geometric version of the Theorem 2.1.

Extension Lemma 4.1. *Let* (M, g) *be a globally Jordan Osserman (resp. globally conformally Jordan Osserman) manifold of signature* $(2, 2)$. *Then for each point of* M *there exists a smooth local pseudo-orthonormal frame* $\{e_1, e_2, e_3, e_4\}$ *with respect to which the matrix representation of the Jacobi operator* $J_R(e_1)$ *(resp. conformal Jacobi operator* $J_W(e_1)$) *is of one of the four types listed in Theorem 2.1.*

Proof. In what follows we will let A denote the Riemann or the Weyl curvature tensor, depending on whether we are working in the conformal setting.

Consider a (smooth) unit timelike vector field e_1 defined on a neighborhood $\mathcal{U}_P$ of $P \in M$. For each $Q \in \mathcal{U}_P$ there exists a pseudo-orthonormal basis $\mathcal{E}_Q = \{e_2, e_3, e_4\}$ of $\{e_1\}^\perp \subset T_Q M$ such that the matrix representation L of $J_A(e_1)$ with respect to $\mathcal{E}_Q$ is independent of Q and is of one of the four types listed in the Theorem 2.1. A priori we do not know if $\mathcal{E}_Q$ is smooth in Q.

Let $\mathcal{F} := \{x_2, x_3, x_4\}$, with x_2 timelike, be a smooth pseudo-orthonormal frame for the sub-bundle $\{e_1\}^{\perp} \subset TM\big|_{\mathcal{U}_P}$. Let X_Q be the change of basis matrix at $Q \in \mathcal{U}_P$:

$$[e_2, e_3, e_4] = [x_2, x_3, x_4] \cdot X_Q.$$

As in the case of $\mathcal{E}_Q$ it is not clear if X_Q is smooth in Q. Let L_Q be the matrix representation of the Jacobi operator $J_A(e_1)$ with respect to the frame $\mathcal{F}$ at a point $Q \in \mathcal{U}_P$; the entries of L_Q depend smoothly on Q. Note that

(4.a) $$X_Q L = L_Q X_Q \quad \text{and} \quad X_Q^T G X_Q = G,$$

where $G = \operatorname{diag}(-1, 1, 1)$. Without loss of generality we will assume $\mathcal{E}_P = \mathcal{F}$, i.e. $X_P = \operatorname{Id}$ and $L = L_P$.

To prove our Extension Lemma it suffices to show the existence of a solution X_Q of the system (4.a) which is smooth in Q (on a neighborhood of P) and which satisfies $X_P = \operatorname{Id}$.

We start with the first of our two equations. Consider the family of linear transformations

$$T_Q : Y \mapsto YL - L_Q Y, \quad Q \in \mathcal{U}_P$$

on the vector space of 3×3 matrices. The dimension of $\ker T_Q$ is independent of Q due to

$$T_Q(Y) = YL - X_Q L X_Q^{-1} Y = X_Q\big(X_Q^{-1} YL - LX_Q^{-1}Y\big) = X_Q \cdot T_P(X_Q^{-1}Y)$$

and the fact that multiplications by invertable matrices are linear isomorphisms. Set

$$k := \dim \ker T_Q = \dim \ker(\mathrm{ad}L),$$

where $\mathrm{ad}L = [L, .]$. By Cramer's Rule the 9×9 system of equations

(4.b) $$X_Q \cdot L - L_Q \cdot X_Q = 0$$

has a k-parameter family of solutions $X_Q = X_Q(\vec{\lambda})$. Since the coefficients of the system vary smoothly with Q the solutions $X_Q(\vec{\lambda})$ depend smoothly on Q and linearly on $\vec{\lambda}$. Note that we can always find parameters $\vec{\lambda}_0$ so that $X_P(\vec{\lambda}_0) = \operatorname{Id}$.

Our next step is to use the second equation of (4.a) to solve for (some of) the parameters $\vec{\lambda}$ in the form of smooth functions of Q. We will accomplish this via the Implicit Function Theorem.

Let $\operatorname{Symm}_G = \{S \mid GS^T G = S\}$. Consider the function

$$F : \mathcal{U}_P \times \mathbb{R}^k \to \ker(\mathrm{ad}L) \cap \operatorname{Symm}_G$$

given by

$$F : (Q, \vec{\lambda}) \mapsto GX_Q(\vec{\lambda})^T G X_Q(\vec{\lambda}).$$

This function is well-defined, i.e. $\operatorname{Im} F \subseteq \ker(\mathrm{ad}L) \cap \operatorname{Symm}_G$, due to a straightforward computation involving identities $LG = GL^T$ and $L_Q G = GL_Q^T$.

Recall that $X_Q(\vec{\lambda})$ depends linearly on $\vec{\lambda}$ and that $X_P(\vec{\lambda}_0) = \operatorname{Id}$. This means that the linearization of F with respect to $\vec{\lambda}$ at $(P, \vec{\lambda}_0)$ is

(4.c) $$\mathcal{L} : \vec{\lambda} \mapsto GX_P(\vec{\lambda})^T G + X_P(\vec{\lambda}).$$

By the Implicit Function Theorem it suffices to prove that the map (4.c) is onto. To understand this map's rank and nullity note that $\vec{\lambda} \mapsto X_P(\vec{\lambda})$ is an isomorphism between $\mathbb{R}^k$ and $\ker(\mathrm{ad}L)$. With this in mind it is clear that

$$\dim \ker \mathcal{L} = \dim \Big(\ker(\mathrm{ad}L) \cap \mathfrak{so}_G \Big),$$

where $\mathfrak{so}_G = \{S \mid GS^TG = -S\}$.

A short computation involving $LG = GL^T$ shows that if $X \in \ker(\mathrm{ad}L)$ then also $GX^TG \in \ker(\mathrm{ad}L)$. Since $X = \frac{1}{2}(X - GX^TG) + \frac{1}{2}(X + GX^TG)$ we have that

$$\ker(\mathrm{ad}L) = \Big(\ker(\mathrm{ad}L) \cap \mathfrak{so}_G \Big) \oplus \Big(\ker(\mathrm{ad}L) \cap \mathrm{Symm}_G \Big).$$

Consequently,

$$k = \dim \ker(\mathrm{ad}L) = \dim \Big(\ker(\mathrm{ad}L) \cap \mathfrak{so}_G \Big) + \dim \Big(\ker(\mathrm{ad}L) \cap \mathrm{Symm}_G \Big)$$

and $k - \dim \ker \mathcal{L} = \dim \big(\ker(\mathrm{ad}L) \cap \mathrm{Symm}_G \big)$. Since $\mathrm{Im}\, F \subseteq \ker(\mathrm{ad}L) \cap \mathrm{Symm}_G$ the linear map (4.c) is onto. Applying the Implicit Function Theorem we obtain (some of) the parameters $\vec{\lambda}$ as smooth functions of Q; this yields to a local smooth solution X_Q of (4.a). The proof of our Extension Lemma is now complete. $\qquad\square$

The proof of the last of our results, Theorem 1.5, is immediate from the Extension Lemma and the proof of Proposition 1.2. One simply needs to replace the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ with the smooth local pseudo-orthonormal frame given by the Extension Lemma.

5. Related problems

We conclude our paper with two related open problems.

(1) Modified Clifford algebraic curvature tensor is always (both spacelike and timelike) Jordan Osserman. It is known that in certain dimensions and signatures (such as signature (2,2) presented here) the converse also holds: a (Jordan) Osserman algebraic curvature tensor allows a modified Clifford structure. Is this a phenomenon which holds in general?

(2) Corollary 2.2 is the crucial background result which made our approach successful. Is there a higher dimensional analogue of Corollary 2.2? Such a result would prove useful in answering the question raised above.

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PG: Mathematics Department, University of Oregon, Eugene Or 97403 USA. Email: *gilkey@darkwing.uoregon.edu*

SN: Mathematical Institute, Sanu, Knez Mihailova 35, p.p. 367, 11001 Belgrade, Serbia. Email: *stanan@mi.sanu.ac.yu*

IS: Department of Mathematical Sciences, Lewis and Clark College, 0615 SW Palatine Hill Road, MSC 110, Portland, Oregon, 97219 USA. Email: *istavrov@lclark.edu*